

# A Note on Segre Types of Second Order Symmetric Tensors in 5-D Brane-world Cosmology

M.J. Rebouças\* and J. Santos†

\* Centro Brasileiro de Pesquisas Físicas

Rua Dr. Xavier Sigaud 150

22290-180 Rio de Janeiro – RJ, Brazil

† Universidade Federal do Rio Grande do Norte

Departamento de Física, Caixa Postal 1641

59072-970 Natal – RN, Brazil

February 7, 2008

## Abstract

Recent developments in string theory suggest that there might exist extra spatial dimensions, which are not small nor compact. The framework of most brane cosmological models is that in which the matter fields are confined on a brane-world embedded in five dimensions (the bulk). Motivated by this we reexamine the classification of the second order symmetric tensors in 5-D, and prove two theorems which collect together some basic results on the algebraic structure of these tensors in 5-dimensional space-times. We also briefly indicate how one can obtain, by induction, the classification of symmetric two-tensors (and the corresponding canonical forms) on  $n$ -dimensional ( $n > 4$ ) spaces from the classification on 4-dimensional spaces. This is important in the context of 11-D supergravity and 10-D superstrings.

---

\*reboucas@cbpf.br

†janilo@dfte.ufrn.br

# 1 Introduction

The idea of extra dimensions has a long and honourable history that goes back to the works of Nordström, Kaluza and Klein [1] – [3]. Since that time Kaluza-Klein-type theories in five or more dimensions have been used and reemerged over the years in several contexts.

In gauge theories they have been employed in the quest for unification of the fundamental interactions in physics. In this case the major idea is that the various interactions in nature might be unified by enlarging the dimensionality of the space-time. Kaluza and Klein themselves studied how one could unify Einstein’s theory of gravitation and Maxwell’s theory of electromagnetism in a five-dimensional (5–D) framework. The higher-dimensional Kaluza-Klein approach was later used, especially among those investigating the eleven-dimensional (11–D) supergravity and ten-dimensional (10–D) superstrings.

Recent developments in string theory and its extension  $M$ -theory have suggested an alternative scenario in which the matter fields are confined on 3–D brane-world embedded in  $1 + 3 + d$  dimensions (the bulk). It is not necessary for the  $d$  extra spatial dimensions to be small and compact; and only gravity and other exotic matter such as the dilaton can propagate in the bulk. This general picture can be simplified to a 5–D context where matter fields are restricted to a 4–D space-time while gravity acts in 5–D [4, 5]. In this limited 5–D framework most work in brane-world cosmology has been done (see, for example, [6, 7] and references therein).

A systematic and independent approach to the 5–D non-compact Kaluza-Klein scenario, known as space-time-matter theory (STM), has also been discussed in a number of papers (see, e.g., [8] – [11] and references therein; and also [12] – [15] ). More recently the equivalence between STM and brane-world theories has been discussed [16], and although they have quite different motivation for the introduction of extra dimension they are, in a sense, mathematically equivalent, and solution of STM can be taken over to 5–D brane-world theory (see [16] for details).

In 4–D general relativity (GR), it is well known that the curvature tensor can be uniquely decomposed into three irreducible parts, namely the Weyl tensor (denoted by  $W_{abcd}$ ), the traceless Ricci tensor ( $S_{ab} \equiv R_{ab} - \frac{1}{4}Rg_{ab}$ ) and the Ricci scalar ( $R \equiv R_{ab}g^{ab}$ ). The algebraic classification of the Weyl part of the Riemann tensor, known as Petrov classification, has played a significant role in the study of various topics in GR. However, for full classification of curvature tensor of nonvacuum space-times one also has to consider the Ricci part of the curvature tensor, which by virtue of Einstein’s equations  $G_{ab} =$

$\kappa T_{ab} + \Lambda g_{ab}$  clearly has the same algebraic classification of both the Einstein tensor  $G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab}$  and the energy momentum tensor  $T_{ab}$ .

The Petrov classification [17] has played an important role in the investigation of various issues in general relativity [18]. The algebraic classification of the Ricci part  $S_{ab}$ , known as Segre classification, has been discussed by several authors [19] and is of great importance in three contexts. One is in understanding some purely geometrical features of space-times [20]. The second one is in classifying and interpreting matter field distributions [21]. The third is as part of the procedure for checking whether apparently different space-times are in fact locally the same up to coordinate transformations [22] – [25] (for examples of the use of this invariant characterization in a class of Gödel-type space-times [26] see [27]).

In 1995 Santos *et al.* [28] studied the algebraic classification of second order symmetric tensors defined on 5–D locally Lorentzian manifolds  $M$ . Their analysis is made from first principles, i.e., without using the previous classifications on lower dimensional space-times. However, as concerns the classification itself their approach is by no means straightforward.

In a subsequent paper Hall *et al.* [29] reexamined the algebraic classification of second order symmetric tensor in 5–D space-times under different premises. They have used the knowledge of the algebraic classification of symmetric two-tensors in 4–D space-times as a given starting point, and then shown that the 5–D algebraic classification obtained in [28] could be achieved in a considerably simpler way. A set of canonical forms for  $T_{ab}$  was also presented. In [29] they stated *with no proof* a theorem which collects together a number of important properties of the algebraic structure of the symmetric two-tensors in 5–D. The main goal of the present paper is to offer a detailed proof of that theorem, and to call the attention to the relevance of the algebraic classification of an arbitrary second order symmetric tensor (hereafter denoted by  $T_{ab}$ ) in the context of 5–D brane-worlds. We also briefly indicate how one can obtain, by induction, the classification of symmetric two-tensors (and the corresponding canonical forms) on  $n$ -dimensional ( $n > 4$ ) spaces from the classification on 4-dimensional spaces. This is important in the context of 11–D supergravity and 10–D superstrings.

To make this paper as clear and selfcontained as possible, in the next section we introduce the notation that will be used, and present a brief summary of the main results on Segre classification in 5–D obtained in our previous articles [28, 29]. These results are required for section 3, where we prove two theorems, related to the algebraic structure of

a second order symmetric two-tensor  $T$  defined in a 5-D locally Lorentzian manifold  $M$ .

To close this section we note that the results of this paper apply to any second order real symmetric tensor (such as Einstein, Ricci and energy momentum) defined on 5-D locally Lorentzian manifolds.

## 2 Preliminaries

In this section we shall fix our general setting, define the notation and briefly review the main results regarding the classification of a second order symmetric tensor in 5-D, required for the last section.

The algebraic classification of symmetric two-tensors  $T$  at a point  $p \in M$  can be cast in terms of the eigenvalue problem

$$T^a_b V^b = \lambda \delta^a_b V^b , \quad (2.1)$$

where  $\lambda$  is a scalar,  $V^b$  is a vector and the mixed form  $T^a_b$  of the tensor  $T$  may be thought of as a linear operator  $T : T_p(M) \longrightarrow T_p(M)$ . Throughout this work  $M$  is a real 5-dimensional space-time manifold locally endowed with a Lorentzian metric of signature  $(- + + + +)$ ,  $T_p(M)$  denotes the tangent vectorial space at a point  $p \in M$  and Latin indices range from 0 to 4, unless otherwise stated. Although the matrix  $T^a_b$  is real, the eigenvalues  $\lambda$  and the eigenvectors  $V^b$  are often complex. A mathematical procedure used to classify matrices in such a case is to reduce them through similarity transformations to canonical forms over the complex field. Among the existing canonical forms the Jordan canonical form (JCF) turns out to be the most appropriate for a classification of  $T^a_b$ .

A basic result in the theory of JCF is that given an  $n$ -square matrix  $T$  over the complex field, there exist nonsingular matrices  $X$  such that

$$X^{-1} T X = J , \quad (2.2)$$

where  $J$ , the JCF of  $T$ , is a block diagonal matrix, each block being of the form

$$J_r(\lambda_k) = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_k \end{bmatrix} . \quad (2.3)$$

Here  $r$  is the dimension of the block and  $\lambda_k$  is the  $k$ -th root of the characteristic equation  $\det(T - \lambda I) = 0$ . Hereafter  $T$  will be the real matrix formed with the mixed components  $T^a_b$  of a second order symmetric tensor.

A Jordan matrix  $J$  is uniquely defined up to the ordering of the Jordan blocks. Further, regardless of the dimension of a specific Jordan block there is one and only one independent eigenvector associated to it.

In the Jordan classification two square matrices are said to be equivalent if similarity transformations exist such that they can be brought to the same JCF. The JCF of a matrix gives explicitly its eigenvalues and makes apparent the dimensions of the Jordan blocks. However, for many purposes a somehow coarser classification of a matrix is sufficient. In the Segre classification, for example, the value of the roots of the characteristic equation is not relevant — only the dimension of the Jordan blocks and the degeneracies of the eigenvalues matter. The Segre type is a list  $\{r_1 r_2 \cdots r_m\}$  of the dimensions of the Jordan blocks. Equal eigenvalues in distinct blocks are indicated by enclosing the corresponding digits inside round brackets. Thus, for example, in the degenerated Segre type  $\{(31)1\}$  four out of the five eigenvalues are equal; there are three linearly independent eigenvectors, two of which are associated to Jordan blocks of dimensions 3 and 1, and the last eigenvector corresponds to a block of dimension 1.

In classifying symmetric tensors in a Lorentzian space two refinements to the usual Segre notation are often used. Instead of a digit to denote the dimension of a block with complex eigenvalue a letter is used, and the digit corresponding to a timelike eigenvector is separated from the others by a comma.

In this work we shall deal with one type of pentad of vectors, namely the semi-null pentad basis  $\{\mathbf{l}, \mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$ , whose non-vanishing inner products are only

$$l^a m_a = x^a x_a = y^a y_a = z^a z_a = 1 . \quad (2.4)$$

At a point  $p \in M$  the most general decomposition of  $T_{ab}$  in terms of semi-null pentad basis for symmetric tensors at  $p \in M$  is manifestly given by [28]

$$\begin{aligned} T_{ab} = & \sigma_1 l_a l_b + \sigma_2 m_a m_b + \sigma_3 x_a x_b + \sigma_4 y_a y_b + \sigma_5 z_a z_b + 2 \sigma_6 l_{(a} m_{b)} \\ & + 2 \sigma_7 l_{(a} x_{b)} + 2 \sigma_8 l_{(a} y_{b)} + 2 \sigma_9 l_{(a} z_{b)} + 2 \sigma_{10} m_{(a} x_{b)} + 2 \sigma_{11} m_{(a} y_{b)} \\ & + 2 \sigma_{12} m_{(a} z_{b)} + 2 \sigma_{13} x_{(a} y_{b)} + 2 \sigma_{14} x_{(a} z_{b)} + 2 \sigma_{15} y_{(a} z_{b)} , \end{aligned} \quad (2.5)$$

where the coefficients  $\sigma_1, \dots, \sigma_{15} \in \mathbb{R}$ .

We can now present the classification of a symmetric two-tensor in 5-D by recalling the following theorem, proved in [28, 29]:

**Theorem 1** *Let  $M$  be a real five-dimensional manifold endowed with a Lorentzian metric  $g$  of signature  $(- + + +)$ . Let  $T^a_b$  be the mixed form of a second order symmetric tensor  $T$  defined at any point  $p \in M$ . Then  $T^a_b$  takes one of the following Segre types:  $\{1, 1111\}$ ,  $\{2111\}$ ,  $\{311\}$ ,  $\{z \bar{z} 111\}$ , or some degeneracy thereof.*

For each Segre type of the Theorem 1 a semi-null basis with nonvanishing inner products (2.4) can be introduced at  $p \in M$  such that  $T_{ab}$  as given by (2.5) reduces to one of the following canonical forms [28, 29]:

Segre type	Canonical form
$\{1, 1111\}$	$T_{ab} = 2\rho_1 l_{(a}m_{b)} + \rho_2(l_a l_b + m_a m_b) + \rho_3 x_a x_b + \rho_4 y_a y_b + \rho_5 z_a z_b , \quad (2.6)$
$\{2111\}$	$T_{ab} = 2\rho_1 l_{(a}m_{b)} \pm l_a l_b + \rho_3 x_a x_b + \rho_4 y_a y_b + \rho_5 z_a z_b , \quad (2.7)$
$\{311\}$	$T_{ab} = 2\rho_1 l_{(a}m_{b)} + 2l_{(a}x_{b)} + \rho_1 x_a x_b + \rho_4 y_a y_b + \rho_5 z_a z_b , \quad (2.8)$
$\{z \bar{z} 111\}$	$T_{ab} = 2\rho_1 l_{(a}m_{b)} + \rho_2(l_a l_b - m_a m_b) + \rho_3 x_a x_b + \rho_4 y_a y_b + \rho_5 z_a z_b , \quad (2.9)$

and the twenty-two degeneracies thereof. Here  $\rho_1, \dots, \rho_5 \in \mathbb{R}$  and  $\rho_2 \neq 0$  in (2.9).

To close this section we recall that the  $r$ -dimensional ( $r \geq 2$ ) subspaces of  $T_p(M)$  can be classified according as they contain more than one, exactly one, or no null independent vectors, and they are respectively called timelike, null and spacelike  $r$ -subspaces of  $T_p(M)$ . Spacelike, null and timelike  $r$ -subspaces contain, respectively, only spacelike vectors, null and spacelike vectors, or all types of vectors.

### 3 Main Results and Concluding Remarks

In this section we will prove **Theorem 4.1** stated in [29] without a proof. To make the presentation simpler we have divided it into two theorems: one that deals with eigenvectors of  $T^a_b$ , and another that treats the invariant subspaces of  $T^a_b$ .

**Theorem 2** *Let  $M$  be a real 5-dimensional manifold endowed with a Lorentzian metric  $g$  of signature  $(- + + +)$ . Let  $T^a_b$  be the mixed form of a second order symmetric tensor  $T$  defined at a point  $p \in M$ . Then*

Table 1: Segre types, eigenvectors and associated eigenvalues.

Segre types	eigenvectors	eigenvalues
$\{1, 1111\}$	$\mathbf{l} - \mathbf{m}, \mathbf{l} + \mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{z}$	$\rho_1 - \rho_2, \rho_1 + \rho_2, \rho_3, \rho_4, \rho_5$
$\{2111\}$	$\mathbf{l}, \mathbf{x}, \mathbf{y}, \mathbf{z}$	$\rho_1, \rho_3, \rho_4, \rho_5$
$\{311\}$	$\mathbf{l}, \mathbf{y}, \mathbf{z}$	$\rho_1, \rho_4, \rho_5$
$\{z \bar{z} 111\}$	$\mathbf{l} - i\mathbf{m}, \mathbf{l} + i\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{z}$	$\rho_1 - i\rho_2, \rho_1 + i\rho_2, \rho_3, \rho_4, \rho_5$

- (i)  $T_b^a$  has a timelike eigenvector if and only if it is diagonalable over  $\mathbb{R}$  at  $p$ .
- (ii)  $T_b^a$  has at least three real orthogonal independent eigenvectors at  $p$ , two of which (at least) are spacelike.
- (iii)  $T_b^a$  has all eigenvalues real at  $p$  and is not diagonalable if and only if it has an unique null eigendirection at  $p$ .
- (iv) If  $T_b^a$  has two linearly independent null eigenvectors at  $p$  then it is diagonalable over  $\mathbb{R}$  at  $p$  and the corresponding eigenvalues are real.

### Proof

- (i) In the previous section (see also [28, 29]) we have shown that if  $T_b^a$  admits a timelike vector it is diagonalable. Reciprocally, if  $T_b^a$  is a diagonal real matrix one writes down a general real symmetric matrix  $g_{ab}$ , uses that  $g_{ac}T_b^c$  is symmetric and obtains  $g_{ab} = \text{diag}(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$ , where  $\mu_1, \dots, \mu_5 \in \mathbb{R}$ . As  $\det g_{ab} < 0$ , then  $\mu_a \neq 0$  ( $a = 1, \dots, 5$ ). As the metric  $g$  has signature  $(- + + +)$  one and only one  $\mu_a$  is negative. The scalar product on  $T_p(M)$  is defined by  $g_{ab}$ , then the norms of the eigenvectors of  $T_b^a$  have the same sign as  $\mu_a \neq 0$ . Since one  $\mu_a < 0$ , one of the eigenvectors must be timelike.
- (ii) From (2.6) – (2.9) one can easily find the complete sets of linearly independent eigenvectors and associated eigenvalues for each non-degenerated Segre type. For each type the orthogonality is ensured by (2.4). In Table 1 we present the eigenvectors and the corresponding eigenvalues for each Segre type. For all types there are at least three orthogonal eigenvectors, two of which are spacelike. We notice that

degeneracies between eigenvalues give rise to eigenspaces (subspaces containing only eigenvectors), in which the number of eigendirections increases.

- (iii) The non-diagonable, non-degenerated Segre types of  $T_b^a$  are  $\{2111\}$  and  $\{311\}$ .

From (ii) we learn that both have only one null eigenvector  $\mathbf{l}$ . Even when there are degeneracies involving the eigenvalues associated to the null eigenvector for these types we still have only one null direction invariant under  $T_b^a$ . The converse is clear if one notes that besides the types  $\{2111\}$  and  $\{311\}$  the possible types in which one can have null eigenvectors are the degeneracies of the type  $\{1, 1111\}$  involving the eigenvalue associated to the timelike eigenvector. However, even in the simplest degenerated type  $\{(1, 1)111\}$  there exist *two* null independent eigendirections. So, these degenerated types should not be considered.

- (iv) Let  $\mathbf{k}$  and  $\mathbf{n}$  be the two linearly independent null eigenvectors of  $T_b^a$  at  $p$  with respective eigenvalues  $\mu$  and  $\nu$ . The symmetry of  $T_{ab}$  implies that  $\mu = \nu$ . Moreover,  $\mathbf{k} + \mathbf{n}$  and  $\mathbf{k} - \mathbf{n}$  are also linearly independent eigenvectors, one of which can easily be shown to be timelike. Therefore, from (i)  $T_b^a$  is necessarily diagonable.

**Theorem 3** *Let  $M$  be a real 5-dimensional manifold endowed with a Lorentzian metric  $g$  of signature  $(- + + +)$ . Let  $T_b^a$  be the mixed form of a second order symmetric tensor  $T$  defined at a point  $p \in M$ . Then*

- (i) *There always exists a 2-D spacelike subspace of  $T_p(M)$  invariant under  $T_b^a$ .*
- (ii) *If a non-null subspace  $\mathcal{V}$  of  $T_p(M)$  is invariant under  $T_b^a$ , then so is its orthogonal complement  $\mathcal{V}^\perp$ .*
- (iii) *There always exists a 3-D timelike subspace of  $T_p(M)$  invariant under  $T_b^a$ .*
- (iv)  *$T_b^a$  admits a  $r$ -dimensional ( $r = 2, 3, 4$ ) null invariant subspace  $\mathcal{N}$  of  $T_p(M)$  if and only if  $T_b^a$  has a null eigenvector, which lies in  $\mathcal{N}$ .*

## Proof

- (i) Indeed, the spacelike eigenvectors  $\mathbf{y}$  and  $\mathbf{z}$  referred to in the item (ii) of the above theorem 2 span a 2-D spacelike subspace of  $T_p(M)$  invariant under  $T_b^a$ .

- (ii) Let  $r$  be the dimension of the non-null subspace  $\mathcal{V}$ , and  $\mathbf{e}_i$  ( $i = 1, \dots, r$ ) a basis of it. Let  $\mathbf{e}_\mu^\perp$  ( $\mu = r+1, \dots, 5$ ) be a basis of  $\mathcal{V}^\perp$ . As  $\mathcal{V}$  is invariant under  $T_b^a$  the vectors  $T \mathbf{e}_i \in \mathcal{V}$ , and therefore  $\mathbf{e}_\mu^\perp \cdot T \mathbf{e}_i = 0$ , which together with the symmetry of  $T_{ab}$  leads to  $\mathbf{e}_i \cdot T \mathbf{e}_\mu^\perp = 0$ . Then for any vector  $\mathbf{v} \in \mathcal{V}^\perp$  we have  $T \mathbf{v} \in \mathcal{V}^\perp$ , i.e.  $\mathcal{V}^\perp$  is also invariant under  $T_b^a$ .
- (iii) Indeed, from (i) and (ii) it follows that the subspace of  $T_p(M)$  spanned by  $\mathbf{l}$ ,  $\mathbf{m}$  and  $\mathbf{x}$  is invariant under  $T_b^a$  given in eqs. (2.6) – (2.9).
- (iv) Clearly every  $r$ -dimensional ( $r = 2, 3, 4$ ) null subspace of  $T_p(M)$  can be spanned by a null vector  $\mathbf{n}$  (say) and a set of  $r - 1$  orthogonal spacelike vectors  $\mathbf{x}_i$  such that  $\mathbf{n} \cdot \mathbf{x}_i = 0$ . Consider the 2-dimensional invariant null subspace generated by such a pair of basis vectors  $\mathbf{n}$  and  $\mathbf{x}$ . Thus

$$T_b^a n^b = \alpha_1 n^a + \alpha_2 x^a, \quad (3.1)$$

$$T_b^a x^b = \beta_1 n^a + \beta_2 x^a, \quad (3.2)$$

where  $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$  and  $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{n} = 0$ . Since  $T_{ab}$  is symmetric we have  $x_a T_b^a n^b = n_a T_b^a x^b$ , which together with (3.1) and (3.2) yields  $\alpha_2 = 0$ , hence  $\mathbf{n}$  is an eigenvector. The proofs for null subspaces of dimension 3 and 4 are similar. The converse is easy to show. Indeed, from eqs. (2.6) – (2.9) one learns that the Segre types which admit null eigenvectors are  $\{2111\}$ ,  $\{311\}$  and their degeneracies, and the type  $\{(1, 1)111\}$  and its further degeneracies. Moreover, from eqs. (2.6) – (2.8) one finds that for all these Segre types  $\mathbf{l}$  is an eigenvector and the subspaces of dimension 2, 3 and 4 spanned by  $\{\mathbf{l}, \mathbf{x}\}$ ,  $\{\mathbf{l}, \mathbf{x}, \mathbf{y}\}$  and  $\{\mathbf{l}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$ , respectively, are null and invariant under  $T_b^a$ .

Before closing this article we remark that by a similar procedure to that used in the lemma 3.1 of [29] one can show that a symmetric two-tensor  $T$  defined on an  $n$ -dimensional ( $n \geq 4$ ) Lorentz space has at least one real spacelike eigenvector. The existence of this eigenvector can be used to reduce, by induction, the classification of symmetric two-tensors on  $n$ -dimensional ( $n > 4$ ) spaces to the classification (and the corresponding canonical forms) on 4-dimensional spaces, thus recovering in a simpler way the Segre types of symmetric two-tensors on  $n$ -dimensions and the corresponding set of canonical forms derived in ref. [30]. This is important in the context of 11–D supergravity and 10–D superstrings.

Finally we mention that it has sometimes been assumed that the source term ( $T_{ab}$  in 4-D) restricted to the brane is a mixture of ordinary matter and a minimally coupled scalar field [31, 32], where the gradient of the scalar field  $\phi_a \equiv \phi_{;a}$  is a timelike vector. In these cases the scalar field can mimic a perfect fluid, i.e., it is of Segre type  $\{(1, (111))\}$ . However, according to ref. [33], depending on the character of the gradient of the scalar field it can also mimic : (i) a null electromagnetic field and pure radiation (Segre type  $\{(211)\}$ , when  $\phi^a$  is a null vector); (ii) a tachyon fluid (Segre type  $\{(1, 11)1\}$ , when  $\phi^a$  is a spacelike vector); and clearly (iii) a  $\Lambda$  term (Segre type  $\{(1, 111)\}$ , when the scalar field is a constant).

## Acknowledgments

M.J. Rebouças thanks J.A.S. Lima for his kind hospitality at Physics Department of Federal University of Rio Grande do Norte (DFTE-UFRN) and A.F.F. Teixeira for reading this manuscript and indication of some omissions. He also acknowledges CNPq for the grant under which this work was carried out.

## References

- [1] G. Nordström, Z. Phys. **15**, 504 (1914).
- [2] Th. Kaluza, Sitzungsberichte preuss. Akad. Wiss. – Phys. Math. Klasse **LIV**, 966 (1921).
- [3] O.Z. Klein, Z. Phys. **37**, 895 (1926).
- [4] P. Horava and Edward Witten, Nucl. Phys. **B460**, 506 (1996);  
 P. Horava and Edward Witten, Nucl. Phys. **B475**, 94 (1996);  
 E. Witten, Nucl. Phys. **B471**, 135 (1996);  
 A. Lukas, B.A. Ovrut, K.S. Stelle and D. Waldram, Phys. Rev. **D59**, 086001 (1999);  
 A. Lukas, B.A. Ovrut, K.S. Stelle and D. Waldram Nucl. Phys. **B552**, 246 (1999);  
 L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999);  
 L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999);  
 T. Shiromizu, K. Maeda, and M. Sasaki, Phys. Rev. D **62**, 024012 (2000).

- [5] K. Akama, hep-th/0001113 (reproduced from 1982 article);  
 V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. **125B**, 139 (1983);  
 M. Visser, Phys. Lett. **B159**, 22 (1985);  
 M. Gogberashvili, Europhys. Lett. **49**, 396 (2000).
- [6] P. Brax and C. van de Bruck, Class. Quantum Grav. **20** R201 (2003).
- [7] D. Langlois, Prog. Theor. Phys. Suppl. **148**, 181 (2003).
- [8] P.S. Wesson and J. Ponce de Leon, J. Math. Phys. **33**, 3883 (1992).
- [9] P.S. Wesson *et al.*, Int. J. Mod. Phys. **A11**, 3247 (1996).
- [10] J.M. Overduin and P.S. Wesson, Phys. Rep. **283**, 303 (1997).
- [11] P.S. Wesson, *Space-Time-Matter*, World Scientific, Singapore (1999).
- [12] F.M. Paiva, M.J. Rebouças and A.F.F. Teixeira, J. Math. Phys. **38**, 4228 (1997).
- [13] M.J. Rebouças and A.F.F. Teixeira, Int. J. Mod. Phys. **A13**, 3181 (1998).
- [14] M.J. Rebouças and A.F.F. Teixeira, J. Math. Phys. **39**, 2180 (1998).
- [15] H.L. Carrion, M.J. Rebouças and A.F.F. Teixeira, J. Math. Phys. **40**, 4011 (1999).
- [16] J. Ponce de Leon, Mod. Phys. Lett. **A16**, 2291 (2001).
- [17] A.Z. Petrov, Sci. Notices of Kazan State University **114**, 55 (1954);  
 A.Z. Petrov, *Einstein Spaces*, Pergamon Press (1969);  
 R. Penrose, Ann. Phys. NY **10**, 171 (1960);  
 R. Penrose and W. Rindler , *Spinors and Space-Time*, vol. 2, Cambridge U. P., Cambridge (1986).
- [18] See, for example, D. Kramer, H. Stephani, M. MacCallum and E. Herlt, *Exact Solutions of Einstein's Equations*, Cambridge U. P., Cambridge (1980).
- [19] G.S. Hall, Diff. Geom. **12**, 53 (1984). This reference contains an extensive bibliography on the classification of the energy momentum tensor on 4-dimensional space-times.

- [20] R.V. Churchill, Trans. Amer. Math. Soc. **34**, 784 (1932);  
 J. Plebański, Acta Phys. Pol. **26**, 963 (1964);  
 W.J. Cormack and G.S. Hall, J. Phys. **A12**, 55 (1979).
- [21] G.S. Hall and D.A. Negm, Int. J. Theor. Phys. **25**, 405 (1986);  
 G.S. Hall, Arab. J. Sci. Eng. **9**, 87 (1984);  
 J.J. Ferrando, J.A. Morales and M. Portilla, Gen. Rel. Grav. **22**, 1021 (1990);  
 M.J. Rebouças and A.F.F. Teixeira, J. Math. Phys. **32**, 1861 (1991);  
 M.J. Rebouças and A.F.F. Teixeira, J. Math. Phys. **33**, 2855 (1992);  
 J. Santos, M.J. Rebouças and A.F.F. Teixeira, J. Math. Phys. **34**, 186 (1993).
- [22] E. Cartan, *Leçons sur la Géométrie des Espaces de Riemann*, Gauthier-Villars, Paris (1951). Reprinted, Éditions Jacques Gabay, Paris, 1988. English translation by J. Glazebrook, Math. Sci. Press, Brookline (1983).
- [23] A. Karlhede, Gen. Rel. Grav. **12**, 693 (1980).
- [24] M.A.H. MacCallum and J. E. F. Skea, “SHEEP: A Computer Algebra System for General Relativity”, in *Algebraic Computing in General Relativity, Lecture Notes from the First Brazilian School on Computer Algebra*, Vol. II, edited by M. J. Rebouças and W. L. Roque. Oxford U. P., Oxford (1994); and references therein.
- [25] M.A.H. MacCallum, “Classifying Metrics in Theory and Practice”, in *Unified Field Theory in More Than 4 Dimensions, Including Exact Solutions*, edited by V. de Sabbata and E. Schmutzler. World Scientific Publishing Co., Singapore (1983).
- [26] M.J. Rebouças and J. Tiomno, Phys. Rev. **D28**, 1251 (1983).
- [27] M.J. Rebouças and J.E. Åman, J. Math. Phys. **28**, 888 (1987);  
 F.M. Paiva, M.J. Rebouças and A.F.F. Teixeira, Phys. Lett. **A126**, 168 (1987);  
 J.B. Fonseca-Neto, M.J. Rebouças and M.A.H. MacCallum, Maths. Comp. Simul. **42**, 739 (1996);  
 J.B. Fonseca-Neto and M.J. Rebouças, Gen. Rel. Grav. **30**, 1301 (1998);  
 J.E. Åman, J.B. Fonseca-Neto, M.J. Rebouças and M.A.H. MacCallum, Class. Quantum Grav. **15**, 1031 (1998).
- [28] J. Santos, M.J. Rebouças and A.F.F. Teixeira, J. Math. Phys. **36**, 3074 (1995).

- [29] G.S. Hall, M.J. Rebouças, J. Santos and A.F.F. Teixeira, Gen. Rel. Grav. **28**, 1107 (1996).
- [30] J. Santos, M.J. Rebouças and A.F.F. Teixeira, Gen. Rel. Grav. **27**, 989 (1995).
- [31] R.J. van den Hoogen, A.A. Coley and Y. He, *Phys. Rev.* **D68**, 023502 (2003).
- [32] R.J. van den Hoogen and J. Ibáñez, *Phys. Rev.* **D67**, 083510 (2003).
- [33] J. Santos, M.J. Rebouças and A.F.F. Teixeira, J. Math. Phys. **34**, 186 (1993).